On the existence of a common zero for two commuting vector fields on a surface (Not for publication)

Pablo Lessa

August 27, 2018

Abstract

We give a new proof of Lima's theorem which states that two commuting vector fields on a closed surface with non-zero Euler characteristic must have a common zero. We also classify all pairs of commuting vector fields with finitely many common zeros which are linearly independent outside of the set of common zeros.

1 Introduction

The purpose of this note is to give a new proof of the following theorem from the 60's due to Elon Lages Lima (see [Lim64b], [Lim63], and [Lim64a]):

Theorem 1 (Lima's Theorem)**.** *Let X and Y be two smooth commuting vector fields on a closed surface S with non-zero Euler characteristic. Then there exists a common zero of X and Y.*

We assume that the vector fields X and Y above are smooth while Lima's original proof goes through for C^1 vector fields.

Our proof is non-elementary but conceptual and not too long. The main idea is to consider the Riemannian metric on the open set where X and Y are linearly independent for which the two fields are orthonormal. Because this Riemannian metric is complete and flat and its tangent bundle is trivial one obtains that each component of this open set is isometric to either the Euclidean plane, a flat cylinder, or a flat torus. In the three cases we can foliate the component by a family of geodesics with constant slope. We show that if there were no common zeros the slopes could be chosen so that one obtains a one dimensional foliation the entire surface, and therefore the Euler characteristic of the surface would necessarily be zero.

To illustrate the ideas further we also classify all pairs of commuting vector fields with only a finite number of common zeros which are always linearly independent on the complement of the set of common zeros. As a corollary we obtain that any pair of commuting vector fields without common zeros on the Klein bottle must have a one dimensional orbit (i.e. an orbit of one field on which the two fields are linearly dependent at all points).

2 A proof of Lima's theorem

2.1 Step 1: Singular foliations

The vector fields X and Y partition the surface S into "accessibility classes", where two points are in the same class if they can be joined by a finite concatenation of curves each of which is tangent either to X or to Y.

General results of Sussmann and Stefan from the 70s (see [Ste80, Theorem 1], and also [Sus73] and [Ste74]) imply that this partition is in fact a "singular foliation" which entails two things:

1. Each class is a smooth injectively immersed submanifold of S, i.e. either a single point, the image of an injective smooth curve, or an open set.
2. Each point p in a one dimensional class is contained in a "flow box", i.e. there exists a diffeomorphism f from $\mathbb{R}^2$ to an open subset of S with $f(0,0) = p$ such that the image of each horizontal line is contained in a single class, and the image of the horizontal line through $(0,0)$ is a connected component of the intersection of $f(\mathbb{R}^2)$ and the class containing p.

The aforementioned results of Sussmann and Stefan are much more general than what is needed in our case (in particular they apply to an arbitrary number of vector fields which may or may not commute on a manifold of arbitrary dimension). An independent and short proof for the case of two commuting vector fields may be given and essentially boils down the flow box theorem for a single vector field (see for example [Bet01, Theorem 6.4]), and the fact that on a curve tangent to one of the fields the other vector field is either tangent at all points or transversal at all points.

We assume from now on that there are no common zeros of X and Y. Hence all accessibility classes are either one or two dimensional. We say a one dimensional class A is adjacent to an open class U if there is a continuous curve $\alpha : [0,1] \to S$ such that $\alpha(0) \in A$ and $\alpha(t) \in U$ for all $t > 0$. By a lamination we mean a collection of one dimensional leaves of a singular foliation whose union is closed. From the singular foliation structure of the partition into accessibility classes one obtains the following:

Lemma 1. *Let X and Y be two smooth commuting vector fields without common zeros on a closed surface S. Then the union of one dimensional accesibility classes is a lamination, and the union of any open accessibility class with its adjacent one dimensional classes is a smooth injectively immersed surface with boundary.*

2.2 Step 2: Natural metrics on open classes

Notice that the flow associated to either X or Y must preserve both vector fields. Hence the open subset of S consisting of points where X and Y are linearly independent must be invariant by both flows. On the other hand if X and Y are dependent on an entire class then that class must have dimension one. Therefore the open accessibility classes are exactly the connected components of the set where X and Y are linearly independent. In view of this we consider the following:

Definition 1 (Natural metric). *Let U be an open accessibility class of two commuting smooth vector fields X and Y on a closed surface S. We define the "natural metric" on U as the unique Riemannian metric for which the fields X and Y are orthornomal.*

We now use apply Cartan's moving frame method (see for example [Nic07, Section 4.2.3] or [Pet16, Chapter 2, Excersice 15]) to calculate the curvature of the natural metric and see that X and Y are paralel.

Let U be an open class, g its natural metric, and ∇ the associated Levi-Civita connection. Since X and Y are g-orthonormal the one form defined by $\omega(Z) = g(\nabla_Z X, Y)$ satisfies $\nabla_Z X = \omega(Z)Y$ and $\nabla_Z Y = -\omega(Z)X$ for all fields Z on U. In particular one obtains

$$0 = [X, Y] = \nabla_X Y - \nabla_Y X = -\omega(X)X - \omega(Y)Y.$$

It follows that $\omega = 0$ on U so that X and Y are paralel (in particular the flow lines of X and Y are geodesics in U). Also since $-d\omega(X, Y)$ is the curvature of g one obtains that (U, g) is flat. Finally, since the flows associated to X and Y are defined for all time in U, it follows that (U, g) is geodesically complete.

We have obtained the following:

Lemma 2. *Let X and Y be two smooth commuting vector fields on a closed surface S and (U, g) be an open accessibility class endowed with its natural metric. Then (U, g) is flat and geodesically complete. Furthermore, the fields X and Y are paralel and in particular their flow lines define two orthogonal families of unit speed geodesics on (U, g).*

An immediate consequence is the following:

Corollary 1. *Any open accessibility class endowed with its natural metric is isometric to either the Euclidean plane, a flat cylinder, or a flat torus.*

Each open class U can be foliated by geodesics with constant slope (i.e. by geodesics tangent to $aX + bY$ for any choice of constants $a, b \in \mathbb{R}$). We wish to show that this can be done in such a way that when the one dimensional classes are added one obtains a foliation of S.

2.3 Step 3: Admissible slopes

Direct calculation of the Lie bracket shows that any smooth vector field on the real line $a(x)\partial_x$ which commutes with the constant field ∂_x must be constant. This shows that on each one dimensional accessibility class there exist constants $a, b \in \mathbb{R}$ such that $aX + bY = 0$ on the class. Also, for each one dimensional class such a pair of constants (a, b) is unique up to multiplication by a real number.

Suppose now that U is an open accessibility class. Because the union of U and its adjacent one dimensional classes is a separable surface with boundary, one obtains that there are at most countably many one dimensional classes adjacent to U. Therefore there exist constants $a, b \in \mathbb{R}$ such that $aX + bY \neq 0$ in U and on all one dimensional classes adjacent to U. This implies (using the flow box theorem) that the flow lines of $aX+bY$ define a smooth one dimensional foliation of the union of U with all its adjacent one dimensional classes.

Assuming there are no common zeros of X and Y the one dimensional classes form a lamination. Any point in S which is on the boundary of U is in the closure of one dimensional classes adjacent to U. It follows that the above foliation of U in fact extends continuously to the closure of U in S.

By choosing such a foliation on each open accessibility class one obtains a continuous foliation of S by smooth curves. Hence we have shown the following:

Lemma 3. *Let S be a closed surface. If there exists a pair of commuting vector fields on S without common zeros, then S admits a continuous one dimensional foliation by smooth curves.*

From this Lima's theorem follows immediately.

3 Commuting flows without one dimensional classes

As an illustration of the same ideas used above to prove Lima's theorem consider the problem of classifying commuting pairs of vector fields on a surface with only a finite number of common zeros, and no one dimensional accessibility classes.

There are three natural examples:

1. Take the two constant vector fields $X(x, y) = (1, 0), Y(x, y) = (0, 1)$ on $\mathbb{R}^2$ and consider the corresponding vector fields on the quotient space $\mathbb{R}^2/(\mathbb{Z}v + \mathbb{Z}w)$ where $v, w \in \mathbb{R}^2$ are linearly independent. This gives two commuting smooth vector fields on a torus without common zeros.

2. In the same situation above consider the fields obtained on the cylinder $\mathbb{R}^2/\mathbb{Z}v$ compactified with two points. This can be done so that the resulting fields extend smoothly to the compactification. In this situation one obtains two commuting vector fields on a sphere with two common zeros.

3. In the same situation above consider the action of the two fields on a one point compactification of $\mathbb{R}^2$ chosen in such a way that the fields extend smoothly to compactified surface. This gives two commuting vector fields on the sphere with a single common zero.

With the same ideas used above we obtain that the above examples are the only ones possible. In particular, among all surfaces, only the torus and the two sphere admit two commuting vector fields with a finite number of zeros and no one dimensional accessibility classes.

Theorem 2. *Let X and Y be two smooth commuting vector fields on a closed surface S with a finite number of common zeros and no one dimensional accessibility classes. Then there exists a diffeomorphism sending the pair X, Y to a pair of fields constructed as in the previous three examples.*

Proof. Since a finite number of points cannot separate S there is a single two dimensional accessibility class U.

From Corollary 1 one obtains an isometry f from U with its natural metric to either $\mathbb{R}^2/(\mathbb{Z}v + \mathbb{Z}w)$ for some linearly independent pair $v, w \in \mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z}v$ for some non-zero $v \in \mathbb{R}^2$, or $\mathbb{R}^2$ (in each case with the metric inherited from the usual Euclidean metric on $\mathbb{R}^2$). Also notice that f may be chosen to take X and Y to constant vector fields $(1, 0)$ and $(0, 1)$ respectively in in all cases (since X and Y are orthonormal and parallel with respect to the adapted metric).

In the first case, since the open accessibility class is also closed it coincides with S and we are done.

In the second case notice that S is diffeomorphic to the compactification of the flat cylinder $\mathbb{R}^2/(\mathbb{Z}v + \mathbb{Z}w)$ with a finite number of points.

There are constants a, b such that $aX + bY$ is sent to the vector v by the isometry f. Hence, all orbits of the field $aX + bY$ in the open class are periodic and have period 1. Since both X and Y equal zero on the complement of the open class, the orbits of $aX + BY$ have diameter which goes to zero when one approaches the complement. It follows that the boundary of the open class has at most two points. It cannot have a single point since the compactification of a cylinder with a single point is not a topological surface. This concludes the second case.

We now turn to the third case where there is a diffeomorphism f from the open class U to $\mathbb{R}^2$ sending X and Y to the fields $(1, 0)$ and $(0, 1)$ respectively.

Since S is compact there is at least one point $p \in S$ not in the open class U. Consider a simple closed curve α_p bounding a neighborhood U_p of p which contains no other common zeros of X and Y. The image curve $\beta_p = f(\alpha_p)$ is a simple closed curve in $\mathbb{R}^2$ and $f(U_p \setminus \{p\})$ must be the unbounded component of the complement of β_p.

However this shows that p is the only common zero of X and Y. To see this pick a common zero q and consder a curve α_q bounding a neighborhood of q containing no other common zero. The unbounded component of the complement of $\beta_q = f(\alpha_q)$ intersects that of β_p, therefore p is in the neighborhood bounded by α_q and it follows that $p = q$.

Hence S is a one point compactification of the open class, which concludes the proof. □

References

[Bet01] David Betounes. *Differential equations: theory and applications with Maple®*. Springer-Verlag–TELOS, New York, 2001. With 1 CD-ROM (Windows, Macintosh and UNIX).

[Lim63] Elon L. Lima. Commuting vector fields on 2-manifolds. *Bull. Amer. Math. Soc.*, 69:366–368, 1963.

[Lim64a] Elon L. Lima. Common singularities of commuting vector fields on 2-manifolds. *Comment. Math. Helv.*, 39:97–110, 1964.

[Lim64b] Elon L. Lima. Commuting vector fields on S^2. *Proc. Amer. Math. Soc.*, 15:138–141, 1964.

[Nic07] Liviu I. Nicolaescu. *Lectures on the geometry of manifolds*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2007.

[Pet16] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.

[Ste74] P. Stefan. Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc. (3)*, 29:699–713, 1974.

[Ste80] P. Stefan. Integrability of systems of vector fields. *J. London Math. Soc. (2)*, 21(3):544–556, 1980.

[Sus73] Héctor J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973.