Two fixed points can force positive entropy of a homeomorphism on a hyperbolic surface (Not for publication)

Pablo Lessa

May 29, 2012

1 Introduction

A common theme in low-dimensional dynamical systems is that the presence of a simple pattern can force the iterates of a function to exhibit very complicated behaviour.

A prime example of this is the fact that a single periodic orbit with period three for a continuous self-map of an interval can force the existence of infinitely many periodic orbits and an explicit form of 'chaos' (see [Šar64] and [LY75]).

For two-dimensional systems the Nielsen-Thurston classification of isotopy classes of homeomorphisms (see [Thu88]) provides a 'simplest' (e.g. minimal entropy[1]) model homeomorphism in each isotopy class. Using this classification several results have been established which show that a small number of periodic orbits with a particular pattern can force a homeomorphism to have positive entropy.

One such result is established in [LM91] where it is shown that three periodic points with non-collinear rotation vectors force a homeomorphism of the two torus which is isotopic to the identity to have positive entropy.

Using homological rotation vectors (which generalize the notion of rotation vectors to surfaces other then the torus) in [Pol92] it is shown that $2g+1$ periodic orbits whose rotation vectors do not lie in a hyperplane suffice to force positive entropy of a homeomorphism isotopic to the identity on a surface of genus g (notice that in this case homology rotation vectors belong to a real vector space of dimension $2g$). This result was later improved in [Mat97] where it is shown that $g+1$ periodic orbits whose rotation vectors form a non-degenerate $g+1$ simplex suffice if $g \geq 2$.

In this article we establish that two fixed points can force positive entropy of a homeomorphism isotopic to the identity on any compact orientable hyperbolic surface. We do this by using the Nielsen-Thurston classification in a way similar to [LM91]. The main novelties are the notion of 'transverse fixed points' (which identifies the pattern necessary to force positive entropy, see Definition 1) and the fact that we are able to deal with the case in which the homeomorphism is 'reducible' in the Nielsen-Thurston classification by showing that one of its irreducible components must be of pseudo-Anosov type.

[1] The word 'entropy' in this article will always refer to 'topological entropy'.

The result is surprising at least in the following two ways:

First, in contrast to the proofs in [Pol92] and [Mat97] our proof is established with out recourse to any generalization of the notion of rotation vectors to other surfaces. In particular there are examples to which our theorem applies which are forced to have positive entropy by a fixed point whose homological rotation vector is zero.

Second, in contrast to the results of [LM91] for the torus we establish that on compact orientable hyperbolic surfaces only two fixed points are necessary to produce entropy instead of three, and this result is independent of the genus of the hyperbolic surface.

As mentioned above the property required of the two fixed points in order to produce entropy is a form of 'transversality'. This property makes sense also for non-periodic orbits and hence poses the problem of whether an analogous result is valid if one replaces the two fixed points by non-periodic trajectories.

In the torus case the results of [Fra89] reduce the non-periodic case to the periodic one by showing that if there are three non-periodic trajectories with non-collinear rotation vectors then there are also three periodic points with non-collinear rotation vectors.

In view of this we pose the following question: If a homeomorphism isotopic to the identity on a compact hyperbolic surface exhibits two transverse non-periodic orbits, does it necessarily also have two transverse periodic orbits? A positive answer to this question would imply a mechanism for obtaining periodic orbits from non-periodic ones and a widely applicable criterion for positive entropy of homeomorphisms isotopic to the identity on a large family of surfaces. Hence we believe this question merits further investigation.

2 Statement

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus. And $f : T^2 \to T^2$ be a homeomorphism which is isotopic to the identity. Fixing a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f the rotation vector of a point $x \in T^2$ is defined as the following limit if it exists:

$$\lim_{n\to+\infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n}$$

where $\tilde{x} \in \mathbb{R}^2$ is any lift of x.

Rotation vectors exist for any periodic orbit and depend on the lift F in a well understood way. In particular the property that there exist three points with non-collinear rotation vectors is independent of the chosen lift F. In this situation the following theorem from [LM91] guarantees that f has positive entropy.

Theorem 1 (Llibre-MacKay). *Let $f : T^2 \to T^2$ be a homeomorphism isotopic to the identity with three fixed points which have non-collinear rotation vectors. Then f has positive entropy.*

Now suppose $S = \mathbb{D}/\Gamma$ is a compact orientable hyperbolic surface (here $\mathbb{D}$ is the Poincaré disk with the usual metric and Γ is a cocompact freely acting group of Möebius transformations) and $f : S \to S$ is a homeomorphism which is isotopic to the identity. We note that all elements of Γ are hyperbolic (i.e. are translations along some geodesic in $\mathbb{D}$).

It can be shown (see Lemma 3 below) that there is a unique lift $F : \mathbb{D} \to \mathbb{D}$ of f which commutes with all elements of Γ.

A fixed point $x \in S$ of f is said to be contractible if there is a lift $\tilde{x}$ of x which is fixed by F.

For any lift $\tilde{x} \in \mathbb{D}$ of a non-contractible fixed point $x \in S$ there exists a non-trivial element $\gamma \in \Gamma$ such that $\gamma^n(\tilde{x}) = F^n(\tilde{x})$ for all $n \in \mathbb{Z}$. It follows that the limits $\lim_{n\to\pm\infty} F^n(\tilde{x})$ exist and are distinct in S^1.

Definition 1 (Transverse fixed points)**.** *Two fixed points $x, y \in S$ are said to be transverse if they are both non-contractible and the following two conditions hold:*

1. *There exist lifts $\tilde{x}, \tilde{y}$ (of x and y respectively) such that the points $\lim_{n\to\pm\infty} F^n(\tilde{x})$ separate the points $\lim_{n\to\pm\infty} F^n(\tilde{y})$ in S^1.*
2. *For any lifts $\tilde{x}, \tilde{y}$ (of x and y respectively) the four points $\lim_{n\to\pm\infty} F^n(\tilde{x})$ and $\lim_{n\to\pm\infty} F^n(\tilde{y})$ are distinct.*

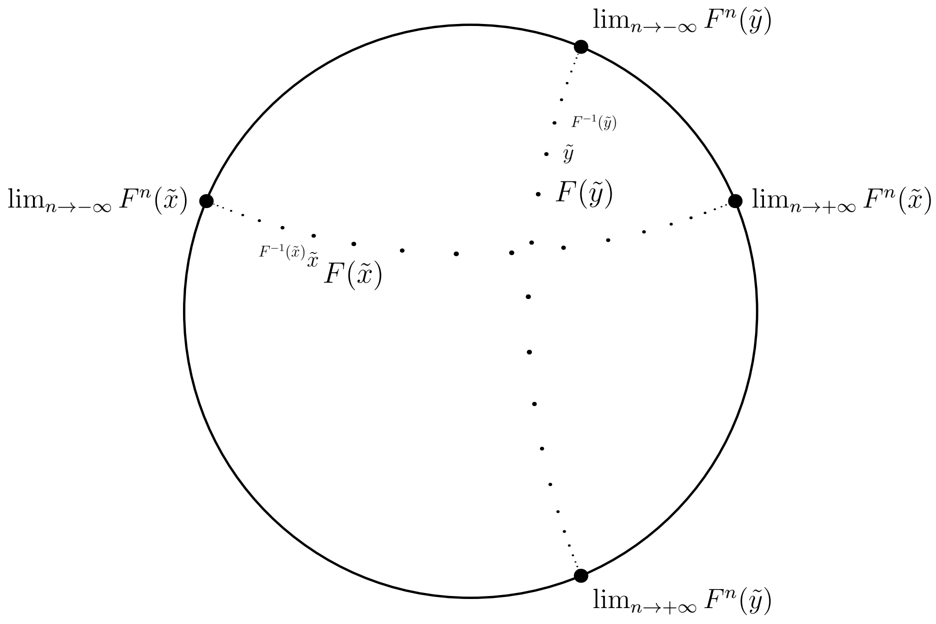

Figure 1: Item 1 of Definition 1.

Another way to look at this definition is to observe that there is a unique closed geodesic freely homotopic to the trajectory of each non-contractible fixed point under the isotopy. Two fixed points are transverse if the closed geodesics associated to each are distinct and intersect (item 2 of the definition forbids the case in which both points are associated to the same self-intersecting geodesic).

The purpose of this article is to prove the following theorem.

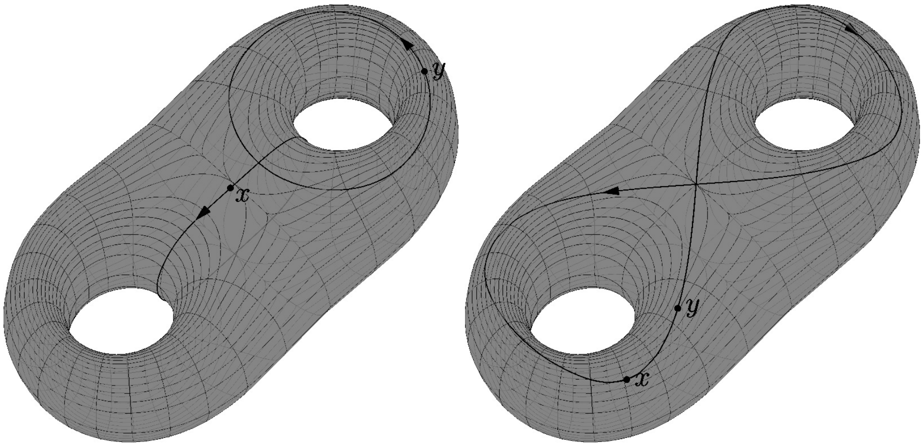

Figure 2: An example and a non-example for Definition 1. Notice that in the rightmost image item 1 of the definition is satisfied but not item 2.

Theorem 2. *Let $f : S \to S$ be a homeomorphism isotopic to the identity which has two transverse fixed points. Then f has positive entropy.*

A first remark is that Theorem 2 requires only two fixed points while Theorem 1 requires three.

Part of this discrepancy can be explained by the fact that there is no canonical way of defining a non-contractible fixed point on T^2 since there is always a lift for which the rotation vector of any particular fixed point is 0.

Correspondingly there is a part of the argument for Theorem 2 which will rely on the uniqueness of the lift F which commutes with all covering transformations and the fact that this lift extends to the boundary of the Poincaré disk as the identity map (Lemma 3). This result has no analog on the two-torus.

Hence Theorem 2 cannot be extended to apply to torus homeomorphisms and doesn't yield an improvement over Theorem 1 in that case (in fact there are examples of homeomorphisms of T^2 with zero entropy and two fixed points with different rotation vectors).

It's easy to obtain examples to which Theorem 2 applies. It suffices to consider for each closed geodesic a homeomorphism supported on a tubular neighborhood which rotates the given geodesic one full turn along itself. The composition of two such homeomorphisms for intersecting geodesics will have a pair of transverse fixed points and hence must have positive entropy.

3 Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the Poincaré disk endowed with the Poincaré metric $\mathrm{d}s^2 = \frac{4}{(1-|z|^2)^2}\mathrm{d}|z|^2$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We will also fix from now on a compact orientable hyperbolic surface $S = \mathbb{D}/\Gamma$ (where Γ is a cocompact and freely acting group of Möebius tranformations, i.e. the group of covering transformations).

Lemma 3 (Canonical lift). *Given $f : S \to S$ isotopic to the identity there is a unique lift $F : \mathbb{D} \to \mathbb{D}$ of f which commutes with all covering transformations. The map F extends continuously to $\overline{\mathbb{D}}$ as the identity on S^1.*

Proof. First we will establish uniqueness. Assume F and G are two lifts of f which commute with all covering transformations. Then $F \circ G^{-1}$ is a covering transformation which commutes with all others. However two hyperbolic Möebius transformations commute if and only if they are translations along the same geodesic, this would imply that Γ is cyclic which is not the case since Γ is cocompact.

To prove that there exists a lift F commuting with all covering transformations take any isotopy $\{f_t : t \in [0,1]\}$ from the identity on S to f and lift it to an isotopy $\{F_t : t \in [0,1]\}$ from the identity on $\mathbb{D}$ to a lift $F = F_1$ of f. We claim F commutes with all covering transformations.

To establish this, fix $\gamma \in \Gamma$ and define for each $t \in [0,1]$ the covering transformation $\gamma_t = F_t^{-1} \circ \gamma^{-1} \circ F_t \circ \gamma$, since $\gamma_t \in \Gamma$ and $t \mapsto \gamma_t$ is continuous it follows that γ_t is constant because Γ is discrete. Furthermore γ_0 is the identity from which we obtain that γ_t is the identity for all t and, setting $t = 1$, that F commutes with γ.

Finally we must prove that F extends to $\overline{\mathbb{D}}$ as the identity on S^1.

We first observe that $p \mapsto d(p, F(p))$ (where $d(p,q)$ denotes the hyperbolic distance between $p, q \in \mathbb{D}$) is continuous and Γ-invariant and therefore bounded (since Γ is cocompact). Next by Proposition 4.3 of [And05] we obtain that $|p - F(p)| \leq C(1 - |p|^2)$ where $C = \sup\{(1 - |F(p)|^2)d(p, F(p)) : p \in \mathbb{D}\}$. From this the result follows. ☐

We will now state the results from Nielsen-Thurston theory (see Sections 7.4 and 7.5 of [FM02]) which we will need to prove Theorem 2. We recall that two homeomorphisms $f, g : S \to S$ are said to be homotopic relative to a finite set $P \subset S$, if there is a homotopy $\{f_t : t \in [0,1]\}$ with $f_0 = f, f_1 = g$ and such that for each t the map f_t fixes all points in P.

Theorem 4 (Nielsen-Thurston pseudo-Anosov component). *If $f : S \to S$ is a homeomorphism fixing all points in a finite set $P \subset S$ then there exists a homeomorphism $g : S \to S$ homotopic to f relative to P and a finite family of simple closed curves $\alpha_1, \ldots, \alpha_r : [0,1] \to S$ such that:*

1. *There is a system of tubular neighborhoods U_i of the curves α_i, which are pairwise disjoint, disjoint from P, and such that there union is g invariant.*
2. *Each component of $S \setminus (P \cup \bigcup_i U_i)$ has negative Euler characteristic*[2].
3. *Either some iterate of g equals the identity on each component of $S \setminus \bigcup_i U_i$ or g has positive entropy.*
4. *The entropy of g is less than or equal to the entropy of f.*

Following [FM02] we call g in the statement above an NT-homeomorphisms for f relative to P and $\alpha_1, \ldots, \alpha_r$ the reducing curves.

[2]The fact that the points of P are to be considered as punctures when calculating the Euler characteristic of a component isn't stated explicitly in Section 7.5 of [FM02] but is clear from the examples on the disk given in Section 7.6 of this reference.

4 Proof of Theorem 2

Let $P = \{x, y\} \subset S$ be the set of two fixed points of f given by hypothesis and $\widetilde{P} \subset \mathbb{D}$ be the set of lifts of these points. Also fix an NT-homeomorphism $g : S \to S$ and a set of reducing curves $\alpha_1, \ldots, \alpha_r$ for f given by Theorem 4.

We will first establish that all the α_i are essential in S. Without loss of generality we may (by replacing f with an iterate) assume that all reducing curves are homotopic relative to P to their images under f (correspondingly each tubular neighborhood U_i given by Theorem 4 is g-invariant).

For the sake of contradiction suppose $\alpha : [0, 1] \to \mathbb{D}$ is a closed curve projecting to a homotopically trivial reducing curve for f. It follows that exactly two lifts $\tilde{x}, \tilde{y}$ of x and y respectively are enclosed by α (otherwise there would be a component of $S \setminus (P \cup \bigcup_i U_i)$ with non-negative Euler characteristic). By assumption $F \circ \alpha$ is homotopic to $\gamma \circ \alpha$ relative to $\widetilde{P}$ for some $\gamma \in \Gamma$.

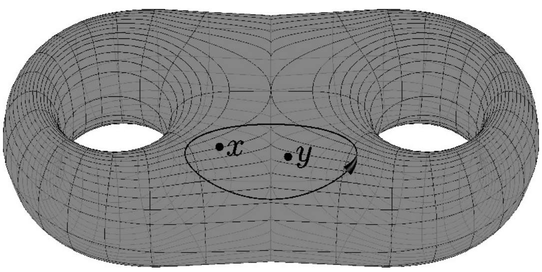

Figure 3: Any non-essential reducing curve must be homotopic to the curve above or its orientation reversed reparametrization relative to $P = \{x, y\}$.

Since F is a homeomorphism $F(\tilde{x})$ and $F(\tilde{y})$ must be enclosed by $F \circ \alpha$. Furthermore, because this curve is homotopic to $\gamma \circ \alpha$ relative to $\widetilde{P}$ one obtains that $\gamma \circ \alpha$ encloses both $F(\tilde{x})$ and $F(\tilde{y})$. By induction it follows that $\gamma^n \circ \alpha$ encloses both $F^n(\tilde{x})$ and $F^n(\tilde{y})$ for all $n \in \mathbb{Z}$. This contradicts the fact that $\lim_{n\to+\infty} F^n(\tilde{x})$ is different from $\lim_{n\to+\infty} F^n(\tilde{y})$.

Next we will show that the reducing curves do not separate x from y in S.

For this purpose suppose $\alpha : \mathbb{R} \to \mathbb{D}$ is a lift of some reducing curve. Since all reducing curves are non-trivial in S it follows that α has two distinct endpoints on S^1. By Lemma 3, $F \circ \alpha$ has the same endpoints on S^1 as α. Because α projects to a reducing curve it must be homotopic relative to $\widetilde{P}$ to a translate $\gamma \circ \alpha$ for some $\gamma \in \Gamma$. Hence the covering transformation γ must fix the endpoints of α and (since α projects to a simple closed curve) we can choose γ to be the identity, so that in fact $F \circ \alpha$ is homotopic relative to $\widetilde{P}$ to α. If $p \in \widetilde{P}$ is on the left of α then, because F preserves orientation, $F(p)$ is on the left of $F \circ \alpha$. Since $F \circ \alpha$ is homotopic to α with a homotopy which doesn't pass through $F(p)$ one obtains that $F(p)$ is to the left of α. Similarly if $p \in \widetilde{P}$ is on the right of α then $F(p)$ is also to the right.

It follows from the preceding paragraph that endpoints of α don't separate the points $\lim_{n\to\pm\infty} F^n(p)$ in S^1 for any $p \in \widetilde{P}$. In particular if $\tilde{x}$ and $\tilde{y}$ are lifts of x and y which are in different components of $\mathbb{D} \setminus \alpha(\mathbb{R})$ then points $\lim_{n\to\pm\infty} F^n(\tilde{x})$ don't separate the points $\lim_{n\to\pm\infty} F^n(\tilde{y})$ in S^1. By property 1 of Definition 1 there must exist lifts $\tilde{x}$ and $\tilde{y}$ which are in the same component of $\mathbb{D} \setminus \alpha(\mathbb{R})$ for all α projecting to a reducing curve. It follows that x and y are in the same component of $S \setminus \bigcup_i U_i$.

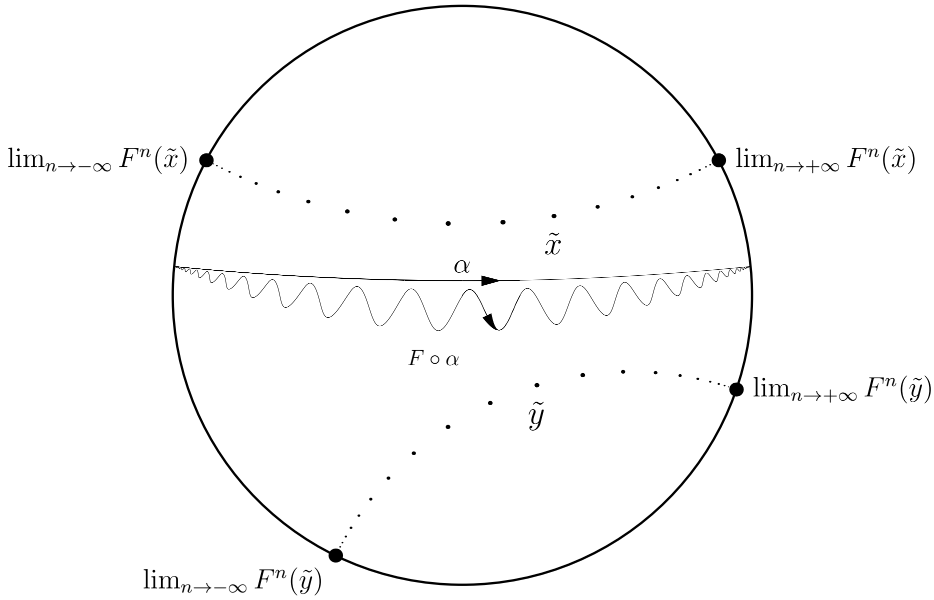

Figure 4: A lift of reducing curve which separates $\tilde{x}, \tilde{y} \in \widetilde{P}$ will separate all their iterates. In this case $\tilde{x}$ and $\tilde{y}$ do not satisfy item 1 of Definition 1.

Let U be the component of $S \setminus \bigcup_i U_i$ containing x and y. At the beginning of this proof we established that a simple closed curve in U which is non-essential in S and which encloses x and y isn't homotopic relative to P to any of its f-iterates. This implies, since g is homotopic to f relative to P, that no iterate of g restricted to U can be the identity. Hence by Theorem 4 we conclude that g has positive entropy and so does f.

5 Acknowledgements

I would like to thank Alejandro Passeggi, Rafael Potrie, Martín Sambarino, and Juliana Xavier for conversations and suggestions which helped motivate and improve this work. Special thanks are also due to Nancy Guelman, Martín Sambarino, and Juliana Xavier for organizing 'Surfaces in Montevideo 2012' and to the participants of this event for making it such a stimulating enviroment for doing mathematics.

References

[And05] James W. Anderson. *Hyperbolic geometry*. Springer Undergraduate Mathematics Series. Springer-Verlag London Ltd., London, second edition, 2005.

[FM02] John Franks and Michał Misiurewicz. Topological methods in dynamics. In *Handbook of dynamical systems, Vol. 1A*, pages 547–598. North-Holland, Amsterdam, 2002.

[Fra89] John Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.*, 311(1):107–115, 1989.

[LM91] J. Llibre and R. S. MacKay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergodic Theory Dynam. Systems*, 11(1):115–128, 1991.

[LY75] Tien Yien Li and James A. Yorke. Period three implies chaos. *Amer. Math. Monthly*, 82(10):985–992, 1975.

[Mat97] Shigenori Matsumoto. Rotation sets of surface homeomorphisms. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(1):89–101, 1997.

[Pol92] Mark Pollicott. Rotation sets for homeomorphisms and homology. *Trans. Amer. Math. Soc.*, 331(2):881–894, 1992.

[Šar64] O. M. Šarkovs′kiĭ. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.*, 16:61–71, 1964.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.